

ORTHOGONAL MODULAR VARIETIES AND THE MODULI OF DEFORMATION GENERALISED KUMMER VARIETIES

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ABSTRACT. For each component of the moduli space of deformation generalised Kummer varieties, there exists a period map to an orthogonal modular variety. The orthogonal modular variety retains information about the associated moduli space, but comes with a lot of extra structure. For this reason, they are very natural objects to study. In general, orthogonal modular varieties are quasi-projective but admit a toroidal compactification. Here we study some combinatorics concerning the boundary of these compactifications, and the singularities within. We intend to later use these ideas along with the ‘*low-weight cusp form trick*’ to determine the Kodaira dimension of certain orthogonal modular varieties.

1. INTRODUCTION

Definition 1.1. *An irreducible symplectic manifold X is a simply connected compact complex Kähler manifold with the property that $H^0(X, \Omega^2) \cong \mathbb{C}\omega$ for a non-degenerate holomorphic 2-form ω .*

A classical problem in moduli theory is to determine the Kodaira dimension of a moduli space. Moduli spaces of irreducible symplectic manifolds can be related to locally symmetric varieties called *orthogonal modular varieties*. Orthogonal modular varieties are given by quotients of a Hermitian symmetric domain of type IV by a discrete subgroup of $O(2, n)$ and, in the above setting, one can often prove results about the Kodaira dimension of the moduli space by studying the associated modular variety. However, one typically needs a careful understanding of the associated modular variety in order to do so.

In this paper, we consider a number of combinatorial problems related to the geometry of the orthogonal modular varieties associated with the moduli of a class of irreducible symplectic manifolds known as *deformation generalised Kummer manifolds*.

1.1. Moduli of irreducible symplectic manifolds. The results and definitions in this section are well-known and can be found, along with further details, in [GHS13].

If X belongs to one of the four known classes of irreducible symplectic manifolds, then $H^2(X, \mathbb{Z})$ can be endowed with the structure of an even non-degenerate lattice L by means of the Beauville-Bogomolov form. The lattice L has signature $(3, n)$ and will be called the *Beauville* or the *Beauville-Bogomolov lattice* of X .

A *polarisation* \mathcal{L} on X is a choice of ample line bundle and a *polarised irreducible symplectic manifold* is a pair (X, \mathcal{L}) consisting of an irreducible symplectic manifold X and an ample line bundle \mathcal{L} on X .

If \mathcal{L} is a polarisation on X , then its first Chern class $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$ defines an element $h \in L$ and a sublattice $h^\perp \subset L$ which we denote by L_h . The *degree* $2d$ of \mathcal{L} is given by the length h^2 of h in L .

If h is primitive in L , we say that \mathcal{L} is a *primitive polarisation*. We shall assume throughout that all polarisations are primitive.

The *polarisation type* of \mathcal{L} is given by the $O(L)$ orbit of h where $O(L)$ is the orthogonal group of L .

If (X, \mathcal{L}) is a polarised irreducible symplectic manifold, then the tuple $(2m, L, h)$ consisting of the dimension $2m$ and Beauville lattice L of X , and polarisation type of \mathcal{L} is called the *numerical type* of (X, \mathcal{L}) and is denoted by N .

Moduli spaces of irreducible symplectic manifolds of numerical type N exist in the sense of GIT (see §3 of [GHS13]), and are related to quotients of Hermitian symmetric domains of type IV by means of the period map. Indeed, if \mathcal{M}_N is the moduli space parametrising irreducible symplectic manifolds of numerical type N , then one has the following theorem of [GHS13].

Theorem. (*Theorem 3.8 [GHS13]*) *For every component \mathcal{M}'_N of \mathcal{M}_N , there is a finite-to-one dominant morphism*

$$\psi : \mathcal{M}'_N \rightarrow O^+(L, h) \backslash \mathcal{D}_h.$$

Where $O^+(L, h)$ is the subgroup of

$$O(L, h) = \{g \in O(L) \mid gh = h\}$$

consisting of elements of real spinor norm 1, and if Ω_h is the domain

$$\Omega_h = \{[x] \in \mathbb{P}(L_h \otimes \mathbb{C}) \mid (x, x) = 0, (x, \bar{x}) > 0\},$$

the domain \mathcal{D}_h is a connected spinor component of Ω_h preserved by the kernel of the real spinor norm on $O(2, n)$. We note that the group $O(L, h)$ can be regarded as a subgroup of $O(L_h)$.

1.2. Orthogonal modular varieties.

Definition 1.2. *If M is a lattice of signature $(2, n)$ and $\Lambda \leq O(M)$ is a subgroup of finite index, an orthogonal modular variety is a quotient of the form*

$$\Omega_M \backslash \Lambda.$$

(We may sometimes broaden the definition to include cases where Λ is an arithmetic subgroup.)

By the results of Baily-Borel [BB66], orthogonal modular varieties are quasi-projective. They are examples of locally symmetric varieties, and so admit toroidal compactifications by the results of [AMRT10]. As a consequence of Hirzebruch-Mumford proportionality (see [GHS08], for example), they are frequently of general type and, if so, one can use the previous theorem of [GHS13] to conclude that an associated moduli space is also of general type.

A general type result for an orthogonal modular variety \mathcal{F} can often be produced by taking a smooth projective model of \mathcal{F} and constructing pluricanonical forms from spaces of orthogonal modular forms satisfying certain properties (the ‘*low-weight cusp form trick*’).

This technique has been used in a number of different examples, including the case of K3 surfaces.

Theorem. (*Theorem 1 [GHS07]*) *The moduli space \mathcal{F}_{2d} of K3 surfaces with a polarisation of degree $2d$ is of general type for any $d > 61$ and for $d = 46, 50, 54, 57, 58$ and 60 . If $d \geq 40$ and $d \neq 41, 44, 45$ or 47 then the Kodaira dimension of \mathcal{F}_{2d} is non-negative.*

To a large extent, the properties that the modular forms need to satisfy depend on the non-canonical singularities in a compactification of the modular variety. It turns out that if the dimension of the modular variety is sufficiently large ($n \geq 9$ by Theorem 5.25 of [GHS13]) there is a toroidal compactification with only canonical singularities. Non-canonical singularities may, however, occur in smaller dimensions; such as in the case of the orthogonal modular varieties associated with deformation generalised Kummer varieties. One therefore needs a more precise understanding in these cases.

1.3. Overview.

Definition 1.3. *If A is an abelian surface and $A^{[n+1]}$ is the Hilbert scheme parametrising $n+1$ points on A , there is a natural morphism*

$$p : A^{[n+1]} \rightarrow A$$

induced by addition on A . The fibre X defined by $X = p^{-1}(0)$ is called a generalised Kummer manifold.

By the results of Beauville [Bea83], generalised Kummer manifolds are examples of irreducible symplectic manifolds, as are their deformations. Deformations of a generalised generalised Kummer manifold will be called a *deformation generalised Kummer manifolds*.

If X is a deformation of the generalised Kummer manifold constructed above then, by the results of [Rap08], the Beauville lattice (which we denote by $L_{2(n+1)}$ of X is given by the signature $(3, 4)$ lattice

$$L_{2(n+1)} = 3U \oplus \langle -2(n+1) \rangle$$

where

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$3U$ denotes $U \oplus U \oplus U$ and $\langle x \rangle$ denotes the rank 1 lattice generated by a vector of length x .

We shall be considering the orthogonal modular varieties \mathcal{F}_{2p^2} associated with the moduli of polarised deformation generalised Kummer manifolds of dimension 4. For arithmetic reasons, we shall assume that they have a polarisation of ‘split type’ (which will be defined in the next section) and degree $2p^2$ for an odd prime p . We consider the singularities in the boundary of a toroidal compactification, and some related combinatorial problems. The singularities in the interior can be understood by using a very different method which we intend to discuss in a future paper.

2. THE MODULAR GROUP

As mentioned in the introduction, if L_{2n} is the Beauville lattice of a deformation Generalised Kummer manifold, then the modular group $O^+(L_{2n}, h_d)$ can be viewed as a subgroup of $O^+(L_{h_d})$. In this section, we characterise this inclusion. In order to do so, we first classify polarisation types.

2.1. Classification of polarisation types.

Proposition 2.1. *If $h_d \in L_{2n}$ is primitive of length $2d > 0$ with $\text{div}(h_d) = f$. Let $g = \left(\frac{2n}{f}, \frac{2d}{f}\right)$, $w = (g, f)$, $g = wg_1$, $f = wf_1$. Then $2n = fgn_1 = w^2 f_1 g_1 n_1$ and $2d = fgd_1 = w^2 f_1 g_1 d_1$ where $(n_1, d_1) = (f_1, g_1) = 1$.*

- (1) *If g_1 is even then h_d exists if and only if $(d_1, f_1) = (f_1, n_1) = 1$ and d_1/n_1 is a quadratic residue modulo f_1 . Moreover, the number of $\tilde{O}(L_{2(n-1)})$ -orbits of h_d with fixed f is equal to $w_+(f_1)\phi(w_-(f_1)).2^{\rho(f_1)}$ where $w = w_+(f_1)w_-(f_1)$ and $w_+(f_1)$ is the product of all powers of primes dividing (w, f_1) , $\rho(n)$ is the number of prime factors of n and $\phi(n)$ is the Euler function.*
- (2) *if g_1 is odd and f_1 is even or f_1 and d_1 are both odd, then such an h_d exists if and only if $(d_1, f_1) = (t_1, 2f_1) = 1$ and $-d_1/n_1$ is a quadratic residue modulo $2f_1$. The number of $\tilde{O}(L_{2n})$ orbits is equal to $w_+(f_1)\phi(w_-(f_1)).2^{\rho(f_1/2)}$ if f_1 is even. and $w_+(f_1)\phi(w_-(f_1)).2^{\rho(f_1)}$ if f_1 and d_1 are both odd.*
- (3) *If g_1 and f_1 are both odd and d_1 is even, then such an h_d exists if and only if $(d_1, f_1) = (n_1, 2f_1) = 1$, $-d_1/(4t_1)$ is a quadratic residue modulo f_1 and w is odd. The number of $\tilde{O}(L_{2n})$ -orbits of such an h_d is equal to $w_+(f_1)\phi(w_-(f_1)).2^{\rho(f_1)}$.*
- (4) *If $c \in \mathbb{Z}$, determined modulo f satisfies $(c, f) = 1$ and $b = (d + c^2n)/f^2$ then*

$$(h_d)_{L_{2n}}^\perp \cong 2U \oplus B$$

where

$$B = \begin{pmatrix} -2b & c\frac{2n}{f} \\ c\frac{2n}{f} & -2t \end{pmatrix}.$$

Proof. Because their Beauville lattices differ only by a factor of $2E_8(-1)$, the classification for deformation generalised Kummer manifolds is essentially identical to the classification given in Proposition 3.6 of [GHS10] for manifolds of $K3^{[2n]}$ -type. \square

Corollary 2.2. *If $w = 1$ and if there exists a primitive vector $h_d \in L_{2n}$ such that $h_d^2 = 2d$ and $\text{div}(h_d) = f$, then all vectors belong to the same $\tilde{O}(L_{2n})$ -orbit.*

Corollary 2.3. *If $f = 1$, then for any n and d , there is only one $\tilde{O}(L_{2n})$ orbit of primitive vectors h_d with $\text{div}(h_d) = 1$. Moreover, $c = 0$ and so*

$$(h_d)_{L_{2n}}^\perp \cong 2U \oplus \langle -2(n+1) \rangle \oplus \langle -2d \rangle$$

Definition 2.4. *A polarisation determined by a primitive vector $h_d \in L_{2n}$ is called split if $\text{div}(h_d) = 1$ and non-split otherwise. We shall indicate this by writing h_d^s instead of h_d .*

Definition 2.5. We define the lattice $L_{2n,2d}$ by

$$L_{2n,2d} = 2U \oplus \langle -2n \rangle \oplus \langle -2d \rangle$$

and the orthogonal modular variety \mathcal{F}_{2d} by

$$\mathcal{F}_{2d} = \mathcal{D}_{h_{2d}^s} / \mathrm{O}^+(L_{2n}, h_d^s).$$

The orthogonal modular variety \mathcal{F}_{2d} is the modular variety associated with the moduli of deformation generalised Kummer $2n$ -folds with split polarisation of degree $2d$.

2.2. Modular groups. In this subsection we characterise the inclusion $\mathrm{O}(L_6, h_{2d}^s) \leq \mathrm{O}(L_{6,2d})$. We start by recalling some of the lattice theoretic results and definitions we shall need.

If L is an even lattice and L^\vee is the dual lattice $\mathrm{Hom}(L, \mathbb{Z})$, the *discriminant group* (and the associated *discriminant form*) $D(L)$ of L is the finite abelian group $D(L) = L^\vee / L$ with the \mathbb{Q}/\mathbb{Z} -valued bilinear form induced from L^\vee [Nik79].

If $S \subset L$ is a primitive sublattice, then the group $\mathrm{O}(L, S)$ is defined by

$$\mathrm{O}(L, S) = \{g \in \mathrm{O}(L) \mid g|_S \in \tilde{\mathrm{O}}(S)\}.$$

As explained in [Nik79], the inclusion $S \subset L$ defines the series of overlattices

$$S^\perp \oplus S < L < L^\vee < (S^\perp)^\vee \oplus S^\vee.$$

Here, the overlattice $S^\perp \oplus S$ is defined by the isotropic subgroup $H = L / (S^\perp \oplus S)$ and because

$$H = L / (S^\perp \oplus S) < (S^\perp)^\vee / S^\perp \oplus S^\vee / S = D(S^\perp) \oplus D(S),$$

H can be regarded as a subgroup of $D(S^\perp) \oplus D(S)$.

One then has the natural projections $p_S : H \rightarrow D(S)$ and $p_{S^\perp} : H \rightarrow D(S^\perp)$. We can now state the following lemma.

Lemma 2.6. [Nik79] [GHS10] Let S be a primitive sublattice in L . Then $g \in \mathrm{O}(L, S)$ if and only if $g(S) = S$, $\bar{g}|_{D(S)} = \mathrm{id}$ and $\bar{g}|_{p_{S^\perp}(H)} = \mathrm{id}$.

Proposition 2.7. If $d > 2$, the group $\mathrm{O}(L_6, h_d^s) \leq \mathrm{O}(L_{6,2d})$ and

$$\mathrm{O}(L_6, h_{2d}^s) \cong \{g \in \mathrm{O}(L_{6,2d}) \mid \bar{g} = \begin{pmatrix} * & 0 \\ * & 1 \end{pmatrix} \in \mathrm{O}(D(L_{6,2d}))\}$$

Moreover, if p is an odd prime, $\mathrm{O}(L_6, h_{2p^2}^s) \leq \mathrm{O}(L_{6,2})$.

Proof. The first part of the argument is essentially a specialisation of Part (i) of Proposition 3.12 in [GHS10].

We can at once consider $\mathrm{O}(L_6, h_d)$ as a subgroup of $\mathrm{O}((h_d^\perp)_{L_{6,2d}})$ because $\mathrm{O}(L_6, h_d)$ acts on both $\langle h_d \rangle$ and h_d^\perp in $\langle h_d \rangle \oplus \langle h_d \rangle^\perp \leq L_6$, but $\mathrm{O}(L_6, h_d)$ acts trivially on $\langle h_d \rangle$ (as $D(\langle h_d \rangle) \cong C_d \neq C_2$).

If $h_d \in L_6$ is split then, by Lemma 2.1, we can take an $\tilde{\mathrm{O}}(L_6)$ representative of h_d to be $h_d = e_3 + bf_3 = e_3 + df_3 \in U \oplus \langle -6 \rangle$.

Let $k_1 = e_3 - df_3$ and $k_2 = l_6$ where l_6 is a generator of the $\langle -6 \rangle$ factor in $L_6 = 3U \oplus \langle -6 \rangle$. A basis for $(h_d^\perp)^\vee$ is given by $\{e_1, f_1, e_2, f_2, k'_1, k'_2, k'_3\}$ where $k'_1 = \frac{k_1}{2d}$, $k'_2 = \frac{k_2}{6}$ and $k'_3 = \frac{h_d}{2d}$.

Now consider the decomposition

$$\langle h_d \rangle \oplus h_d^\perp < L_6 < L_6^\vee < \langle h_d^\vee \rangle \oplus (h_d^\perp)^\vee$$

where $h_d^\vee = \frac{1}{2d}h_d$ and $h_d^\perp = (h_d)^\perp \subset L_6$ is given by $h_d^\perp \cong 2U \oplus \langle -2d \rangle \oplus \langle -6 \rangle$. A simple calculation shows that the subgroup

$$H = L_6 / (\langle h_d \rangle \oplus h_d^\perp) < D(\langle h_d \rangle) \oplus D(h_d^\perp)$$

is equal to $\langle k'_3 - k'_1, d(k'_1 + k'_3) \rangle \leq L_6 / (\langle h_d \rangle \oplus h_d^\perp)$, and so $p_{h_d^\perp}(H) = \langle k'_1 \rangle$. Therefore, by Lemma 2.6 and because $D(h_d^\perp) = \langle k'_1 \rangle \oplus \langle k'_2 \rangle$, we conclude that

$$O(L_6, h_d) \cong \{g \in O(h_d^\perp) \mid \bar{g}|_{p(H)} = \text{id}\}$$

and we obtain the first part of the claim.

For the second part of the claim, let $L_{6,2p^2}$ have basis $\{e_1, f_1, e_2, f_2, v_1, v_2\}$ and let $L_{6,2}$ have basis $\{e'_1, f'_1, e'_2, f'_2, v'_1, v'_2\}$ where $\{e_i, f_i\}$, $\{e'_i, f'_i\}$ are the standard bases for U and v_1, v'_1 generate the copies of $\langle -6 \rangle$ in $L_{6,2p^2}$ and $L_{6,2}$, respectively; and v_2 and v'_2 generate the copies of $\langle -2p^2 \rangle$ and $\langle -2 \rangle$ in $L_{6,2p^2}$ and $L_{6,2}$, respectively.

We define the embedding $L_{6,2p^2} \leq L_{6,2}$ by

$$(e_1, f_1, e_2, f_2, v_1, v_2) \mapsto (e'_1, f'_1, e'_2, f'_2, v'_1, pv_2)$$

and define the totally isotropic subspace M by $M = L_{6,2} / L_{6,2p^2} \leq D(L_{6,2p^2})$.

The lattice $L_{6,2}$ can be recovered from M by noting that $L_{6,2} = \{x \in L_{6,2p^2}^\vee \mid x \bmod L_{6,2p^2} \in M\}$. Moreover, M is of the form $(0, *) \in D(L_{6,2p^2}) = \langle k'_2 \rangle \oplus \langle k'_1 \rangle$.

An element $g \in O(L_6, h_{2p^2}) \leq O(L_{6,2p^2})$ extends naturally to the element $\hat{g} \in O(L_{6,2p^2}^\vee)$ and because $g(k'_1) = k'_1$, the element \hat{g} preserves M . Therefore, $\hat{g}(L_{6,2}) \leq L_{6,2}$ and so g extends to a unique element in $O(L_{6,2})$. \square

Corollary 2.8. *If p is an odd prime and $h_{2p^2} \in L_6$ is split, then*

$$\tilde{O}^+(L_{6,2p^2}) \leq O^+(L_6, h_{2p^2}).$$

2.3. Index bounds. We next use an idea in [Kon93] to show that $O(L_6, h_{2p^2})$ is of finite index in $O(L_{6,2})$. The argument involves considering the action of $O(L_{6,2})$ on a finite quadratic space.

We shall use some classical results on the orthogonal groups of finite type. These can be found in [Die71], for example, but are stated below for the convenience of the reader.

A non-degenerate quadratic space V over a finite field \mathbb{F}_q of odd characteristic is classified in terms of $\dim V$ and the discriminant $\Delta = \det B \in \mathbb{F}_q^* / (\mathbb{F}_q^*)^2$, where B is the bilinear form on V . If $\dim V = 2m$, then V falls into one of two isomorphism classes depending on the value of ϵ where $\epsilon = (-1)^m \Delta \in \mathbb{F}_q^* / (\mathbb{F}_q^*)^2$. They are:

$$\begin{aligned} V_\epsilon^{2m} &= H_1 \oplus \dots \oplus H_m & \text{if } \epsilon = 1 \\ V_\epsilon^{2m} &= V_\theta \oplus H_1 \oplus H_2 \oplus \dots \oplus H_{m-1} & \text{if } \epsilon = -1. \end{aligned}$$

Here, H_i are hyperbolic planes over \mathbb{F}_p and V_θ is the quadratic space $\langle u, v \rangle_{\mathbb{F}_p}$ whose bilinear form is given by $(u, u) = 1$, $(u, v) = 0$ and $(v, v) = \theta$ for some $-\theta \notin (\mathbb{F}_q^*)^2$. If $\dim V = 2m + 1$, there is only one isomorphism class for V , which is given by $V^{2m+1} = H_1 \oplus \dots \oplus H_m \oplus \langle \theta \rangle$ for some $0 \neq \theta \in \mathbb{F}_q$.

We shall also need to know about the order of $O^+(V)$ for a finite quadratic space V . As in [Die71], these are given by

$$(1) \quad |O^+(V^{2m+1})| = (q^{2m} - 1)q^{2m-1}(q^{2m-2} - 1) \dots (q^2 - 1)q$$

and

$$(2) \quad |O^+(V_\epsilon^{2m})| = (q^{2m-1} - \epsilon q^{m-1})(q^{2m-2} - 1)q^{2m-3} \dots (q^2 - 1)q.$$

We define the finite quadratic space Q_p by

$$Q_p = L_{6,2}/pL_{6,2} \leq L_{6,2p^2}/pL_{6,2}.$$

Our first result is that $O(L_{6,2})$ acts transitively on Q_p . In order to show this, we shall use the following two well-known lemmas to construct elements of $O(L_{6,2})$.

Definition 2.9. Let L be an indefinite lattice. If $e \in L$ is isotropic and $a \in e^\perp \subset L$ then the map on L defined by

$$t(e, a) : v \mapsto v - (a, v)e + (e, v)a - \frac{1}{2}(a, a)(e, v)e$$

is called an Eichler transvection and belongs to the group $\widetilde{SO}^+(L)$ [Eic74] (see also §3 of [GHS09]).

Lemma 2.10. The group $\widetilde{SO}^+(2U)$ is isomorphic to $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$.

Proof. Full details of the proof may be found in §3 of [GHS09]. The isomorphism is defined by mapping $(w, x, y, z) \in U \oplus U$ to $\begin{pmatrix} w & -y \\ z & x \end{pmatrix} \in M_2(\mathbb{Z})$, where (w, x, y, z) is given on the standard basis for $U \oplus U$. The inner product on $U \oplus U$ is defined by the determinant on $M_2(\mathbb{Z})$. An element $(A, B) \in SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ acts on $U \oplus U$ by mapping

$$\begin{pmatrix} w & -y \\ z & x \end{pmatrix} \mapsto A \begin{pmatrix} w & -y \\ z & x \end{pmatrix} B.$$

□

Lemma 2.11. The group $O(L_{6,2})$ acts transitively on hyperplanes of the same type in Q_p .

Proof. Let $\{e_1, f_1, e_2, f_2, v_1, v_2\}$ be a basis for $L_{6,2} = 2U \oplus \langle -6 \rangle \oplus \langle -2 \rangle$ where v_1 and v_2 generate $\langle -6 \rangle$ and $\langle -2 \rangle$, respectively and $\{e_i, f_i\}$ are the standard basis for U . If $w = (w_1, w_2, w_3, w_4, w_5, w_6) \in L_{6,2}$ then the Eichler transvections $t(e_1, v_1)$ and $t(e_1, v_2)$ act as

$$t(e_2, v_1)w = (w_1, w_2, w_3 + 3w_4 + 6w_5, w_4, w_5 + w_4, w_6)$$

and

$$t(e_2, v_2)w = (w_1, w_2, w_3 + w_4 + 2w_6, w_4, w_5, w_6 + w_4).$$

Let $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in L_{6,2}/pL_{6,2}$ be non-zero. We can assume that $x_4 \neq 0$ by (if required) applying $t(e_2, v_1)$ and permuting $\{x_1, x_2, x_3, x_4\}$ by elements in $O(2U)$. Then rescale x so that $x_4 = 1$. After repeated application of $t(e_2, v_1)$ and $t(e_2, v_2)$, the element x can be transformed to an element of the form $(x'_1, x'_2, x'_3, x'_4, 0, 0)$; and so can be identified with an element of $2U$. By using the copy of $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$ in $O(2U)$, then element x can be sent to an element of the form $(r, s, 0, 0, 0, 0)$ and then, after rescaling, to an element of the form $(1, a, 0, 0, 0, 0)$.

Now suppose that $u, v \in L_{6,2}/pL_{6,2}$ are given by $u = (1, a, 0, 0, 0, 0)$ and $v = (1, b, 0, 0, 0, 0)$.

If $ab^{-1} \in (\mathbb{F}_p^*)^2$ then there exists $\mu, \lambda \in \mathbb{F}_p$ such that $(\mu u)^2 = (\lambda v)^2$. Let \hat{u} and \hat{v} be defined by $\hat{u} = \mu u = (u_1, u_2, 0, 0, 0, 0)$ and $\hat{v} = \lambda v = (v_1, v_2, 0, 0, 0, 0)$; and suppose that $\hat{u} - \hat{v} = (r, s, 0, 0, 0, 0)$ is non-zero.

If $d = \gcd(r, s)$ and r_1, r_2, s_1, s_2 are solutions to $r_2 u_1 + r_1 u_2 = d$ and $s_2 v_1 + s_1 v_2 = d$, the elements $u', v', w \in e_1^\perp \cap f_1^\perp \subset L_{6,2}$ are defined by $u' = (r_1, r_2, 0, 0, 0, 0)$, $v' = (s_1, s_2, 0, 0, 0, 0)$, and $w = (\frac{r}{d}, \frac{s}{d}, 0, 0, 0, 0)$ where $r' \equiv r \pmod{p}$ and $s' \equiv s \pmod{p}$.

Over \mathbb{F}_p , we have that $(\hat{u}, u') = d$ and $(\hat{v}, v) = d$ and so the element $t(e_2, v')t(f_2, w)t(e_2, u')$ sends \hat{u} to \hat{v} .

Therefore, $O(L_{6,2})$ is transitive on hyperplanes of the same type in $L_{6,2}/pL_{6,2}$. \square

We can now prove our main theorem.

Theorem 2.12. *The group $O^+(L_6, h_{2p^2}^s)$ is of finite index in $O^+(L_6, h_2^s)$ and*

$$|O^+(L_6, h_2^s) : O^+(L_6, h_{2p^2}^s)| \leq 16(p^5 + p^2).$$

Proof. There are natural homomorphisms from $O(L_{6,2}) \rightarrow O(L_{6,2}/pL_{6,2})$ and by Lemma 2.11, the group $O(L_{6,2})$ acts transitively on Q_p and so $O(L_{6,2}/pL_{6,2})$ also acts transitively on Q_p . The group $O(L_6, h_{2p^2}^s) \leq O(L_{6,2})$ stabilises a hyperplane $\Pi \subset Q_p$ and so, by the Orbit-Stabiliser theorem,

$$|O(L_{6,2}) : \text{stab}_{O(L_{6,2})}(\Pi)| = |O(L_{6,2}/pL_{6,2}) : O(L_{6,2p^2}/pL_{6,2})|$$

and

$$|O^+(L_{6,2}) : \text{stab}_{O^+(L_{6,2})}(\Pi)| = |O^+(L_{6,2}/pL_{6,2}) : O^+(L_{6,2p^2}/pL_{6,2})|.$$

(where we have used the fact that $\text{stab}_{O(L_{6,2}/pL_{6,2})}(\Pi) = O(L_{6,2p^2}/pL_{6,2})$ and the fact that the spinor kernel is of index two in the full orthogonal group.) By Lemma 2.7, $O(L_6, h_{2p^2}^s) \leq O(L_{6,2})$ and so

$$\tilde{O}^+(L_{6,2p^2}) \leq O^+(L_6, h_{2p^2}^s) \leq \text{stab}_{O^+(L_{6,2})} \Pi \leq O^+(L_{6,2p^2}).$$

As $O(D(L_{6,2p^2})) \cong V_4 \oplus C_2 \oplus C_2$ where V_4 is the Klein 4-group,

$$|\text{stab}_{O(L_{6,2})} \Pi : O(L_6, h_{2p^2}^s)| \leq |O(L_{6,2p^2}) : \tilde{O}(L_{6,2p^2})| = 16$$

and therefore

$$\begin{aligned} |O^+(L_{6,2}) : O^+(L_6, h_{2p^2}^s)| &\leq 16 |O^+(L_{6,2}/pL_{6,2}) : O^+(L_{6,2p^2}/pL_{6,2})| \\ &\leq 16 \frac{(p^5 - \epsilon p^2)(p^4 - 1)p^3(p^2 - 1)p}{(p^4 - 1)p^3(p^2 - 1)p} \\ &\leq 16(p^5 + p^2). \end{aligned}$$

\square

3. TOROIDAL COMPACTIFICATIONS

In this section, we let \mathcal{V} denote the orthogonal modular variety \mathcal{D}_M/Λ where M is an even lattice of signature $(2, n)$ and $\Lambda \leq \mathrm{O}(M)$ is a subgroup of finite index.

By the results of [GHS07], it is known that \mathcal{V} admits a toroidal compactification in which all of the non-canonical singularities at the boundary lie in the one dimensional boundary components.

Here we provide estimates for \mathcal{F}_{2p^2} of the number of such boundary components, and also bounds for the number of components of the singular locus in each. We end by producing a classification of the non-canonical singularities that can occur.

We start by explaining how to compactify \mathcal{V} . More details on toroidal compactifications can be found in the monographs of [AMRT10] and [BJ06]. A comprehensive summary may be found in [GHS13], which we follow for the expository part of this section.

3.1. Boundary Components. The domain \mathcal{D}_M can be embedded in its compact dual by means of the Harish-Chandra embedding. Under this embedding, the Baily-Borel compactification \mathcal{D}_M^{BB} of \mathcal{D}_M is defined as the closure of \mathcal{D}_M in \mathcal{D}_M^\vee . It can be decomposed as

$$\mathcal{D}_M^{BB} = \mathcal{D} \sqcup \bigsqcup_{P \in \mathcal{P}} F_P$$

where \mathcal{P} is a set of parabolic subgroups of $\mathrm{O}(2, n)$ associated with certain collections of orthogonal roots.

We define \mathcal{P}_{rat} as the subset of \mathcal{P} consisting of rational parabolic subgroups. These are precisely the subgroups of Γ that preserve a totally isotropic subspace $\Pi \subset M \otimes \mathbb{Q}$. Because M is of signature $(2, n)$, the subspace Π is either an isotropic line or a totally isotropic plane. There is a partial compactification \mathcal{D}^* of \mathcal{D} given by

$$\mathcal{D}_M^* = \mathcal{D}_M \sqcup \bigsqcup_{P \in \mathcal{P}_{rat}} F_P.$$

The group Λ acts on the space \mathcal{D}_M^* , and the Baily-Borel compactification \mathcal{V}^* of \mathcal{V} is defined as the quotient of \mathcal{D}_M^* by Λ . For our applications, the important properties of the Baily-Borel compactification are summarised in the following theorem.

Theorem 3.1. [GHS13] [BB66] *The Baily-Borel compactification $(\mathcal{D}_M/\Lambda)^*$ is an irreducible normal complex projective variety. It contains \mathcal{D}_M/Λ as a Zariski-open subset and can be decomposed as*

$$(\mathcal{D}_M/\Lambda)^* = \mathcal{D}_M/\Lambda \sqcup \bigsqcup_{P \in \mathcal{P}_{rat}} F_P/\Lambda_P$$

where \mathcal{P}_{rat} runs over all the Γ -equivalence classes of parabolic subgroups determining rational boundary components.

If $P \in \mathcal{P}_{rat}$ stabilises an isotropic line, then we say that the boundary component F_P/Λ_P is a *0-dimensional boundary component*; otherwise, a *1-dimensional boundary component*.

At a rational boundary component F , the groups $N(F)$ is defined as the normaliser of F taken in $\mathrm{O}(2, n)$, the group $W(F)$ is the unipotent radical of $N(F)$, and the group $U(F)$ is the centre of $W(F)$, respectively.

Their intersections with Γ will be denoted by $N(F)_{\mathbb{Z}}$, $W(F)_{\mathbb{Z}}$, and $U(F)_{\mathbb{Z}}$, respectively.

3.2. Overview of toroidal compactifications. Here we describe how to construct a toroidal compactification of the orthogonal modular variety \mathcal{V} . The construction we describe is in terms of local patches (corresponding to a partial compactification) in a neighbourhood of each boundary component.

If F is a rational boundary component of \mathcal{V} , we define the domain $\mathcal{D}_M(F)$ by $\mathcal{D}_M(F) = U(F)\mathcal{D}_M \subset \mathcal{D}_M^{\vee}$. Because of the Langlands decomposition of $N(F)$, the domain $\mathcal{D}_M(F)$ can be decomposed as

$$(3) \quad \mathcal{D}_M(F) = F \times V(F) \times U(F)_{\mathbb{C}}$$

where $V(F)$ is the complex vector space $W(F)/U(F)$.

If $\mathcal{D}_M(F)'$ is the quotient space $\mathcal{D}_M(F)' = \mathcal{D}_M(F)/U(F)_{\mathbb{C}}$, the spaces $\mathcal{D}_M(F)$, $\mathcal{D}_M(F)'$ and F are related by the diagram

$$\begin{array}{ccc} \mathcal{D}_M(F) & & \\ \pi_F \downarrow & \searrow \pi'_F & \\ & \mathcal{D}_M(F)' & \\ & \swarrow p_F & \\ & F & \end{array}$$

where π_F , p_F , and π'_F are the natural projections onto F , F , and $\mathcal{D}_M(F)$, respectively.

The space $\pi'_F : \mathcal{D}_M(F) \rightarrow \mathcal{D}_M(F)'$ is a principal homogeneous spaces for $U(F)_{\mathbb{C}}$ and the group $N(F)_{\mathbb{Z}}$ acts on $\mathcal{D}_M(F)$. By restricting to $U(F)_{\mathbb{Z}}$, one obtains the principal fibre bundle

$$(4) \quad \mathcal{D}_M(F)/U(F)_{\mathbb{Z}} \rightarrow \mathcal{D}_M(F)'$$

whose fibre is equal to the algebraic torus $T(F)$ given by $U(F)_{\mathbb{C}}/U(F)_{\mathbb{Z}}$.

The group $U(F)$ contains a cone $C(F)$. By selecting a fan $\Sigma \subset C(F)$ and then replacing the torus $T(F)$ in the bundle of equation (4) with the toric variety $X_{\Sigma(F)}$, one obtains a new bundle over $\mathcal{D}_M(F)$ with fibre $X_{\Sigma(F)}$.

A partial compactification for \mathcal{V} in a neighbourhood of the boundary component F is obtained by taking the closure of $\mathcal{D}_M/U(F)_{\mathbb{Z}}$ in the new bundle, and then taking the quotient by $N(F)_{\mathbb{Z}}$.

The final step in constructing the toroidal compactification $\overline{\mathcal{V}}$ of \mathcal{V} is to glue all of the partial compactifications together by identifying the copies of \mathcal{D}_M/Γ contained in each one. In general, Σ must be chosen to satisfy certain conditions in order for the gluing procedure to work. However, for orthogonal modular varieties, these conditions are automatically satisfied.

3.3. Explicit description of the one dimensional boundary components. We describe the compactification at the one dimensional boundary components explicitly,

as in [Sca87], [Kon93] and [GHS07]. For reasons of notational clarity (and where no confusion should arise), we may sometimes use L to denote the lattice $L_{6,2p^2}$.

Lemma 3.2. *Let $E \leq L_{6,2p^2}$ be a primitive, totally isotropic subspace of rank 2 corresponding to the boundary component F . There exists a \mathbb{Z} -basis $\{v_1, \dots, v_6\}$ of $L_{6,2p^2}$ such that $\{v_1, v_2\}$ is a basis for E and $\{v_1, \dots, v_4\}$ is a basis for E^\perp . The basis can be chosen so that the bilinear form Q has Gram matrix*

$$(5) \quad Q = ((v_i, v_j)) = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C \\ {}^t A & {}^t C & D \end{pmatrix}$$

where B is the form on E^\perp/E and

$$A = \begin{pmatrix} 0 & a_1 \\ a_1 a_2 & 0 \end{pmatrix}.$$

Here a_1 and a_2 are the elementary divisors of the group $D(L_{6,2p^2})/H_E^\perp$. Moreover,

$$(a_1, a_1 a_2) \in \{(1, 1), (1, 2p), (1, 6p)\}.$$

Proof. As E and E^\perp are primitive, the claim about the existence of a basis on which Q assumes the form of Equation (5) is immediate. We next consider the matrix A . By considering the Smith normal form of A , we see that A embeds $\langle v_5, v_6 \rangle$ in the dual $\langle v_5^*, v_6^* \rangle$ and so the elementary divisors of A correspond to the elementary divisors of the abelian group $\langle v_5^*, v_6^* \rangle / \langle v_5, v_6 \rangle$.

If $H_E = E^{\perp\perp}/E \leq D(L_{6,2p^2})$, then $H_E^\perp = \langle v_1^*, \dots, v_4^* \rangle$ in $D(L_{6,2p^2})$ and so $\langle v_5^*, v_6^* \rangle / \langle v_5, v_6 \rangle \cong D(L_{6,2p^2})/H_E^\perp$. We next determine H_E and H_E^\perp . As the lattice E is totally isotropic in $L_{6,2p^2}$, the group H_E is totally isotropic in $D(L_{6,2p^2})$. If $D(L_{6,2p^2})$ is identified with $((-1/6) \oplus (-1/2p^2), C_6 \oplus C_{2p^2})$, then $(x, y) \in D(L_{6,2p^2})$ is isotropic if and only if

$$p^2 x^2 + 3y^2 = 0 \pmod{6p^2}.$$

As $(3, p) = 1$, $p|y$ and so, $p^2 x^2 + 3p^2 y_1^2 = 0 \pmod{6p^2}$ and $x^2 + p y_1^2 = 0 \pmod{6}$.

By considering squares modulo 6, we conclude that $x = 0$ or 3 and that x and y must have different parities. The isotropic elements in $D(L_{6,2p^2})$ are, therefore,

$$(x, y) \in \{(0, 2kp), (3, (2k+1)p) \mid k \in \mathbb{Z}\}.$$

The primitive isotropic subspaces of rank 1 in $D(L_{6,2p^2})$ are generated by $x_1 = (0, 2p)$ and $x_2 = (3, p)$ and the single rank 2 totally isotropic subspace is generated by $\langle x_1, x_2 \rangle$.

If $H_E = \langle x_1 \rangle$,

$$H_E^\perp = \{(a, b) \in D(L_{6,2p^2}) \mid pa + 6b \equiv 0 \pmod{6p}\}$$

and so $p|b$, $6|a$ and $H_E^\perp = \langle (0, p) \rangle \cong C_{2p}$.

If $H_E = \langle x_2 \rangle$,

$$H_E^\perp = \{(a, b) \in D(L_{6,2p^2}) \mid pa + b \equiv 0 \pmod{2p}\}$$

and so $p|b$, $2|(a+b)$ and $H_E^\perp = \langle (1, p), (2, 0) \rangle$. If $y_1 = (1, p)$ and $y_2 = (2, 0)$, we also have the relations $6py_1 = 0$ and $3y_2 = 0$. Therefore, $p(2y_1 - y_2) = 0$. Moreover, because $p \equiv \pm 1 \pmod{6}$, $2py_1 = \pm y_2$ and so $H_E^\perp = \langle y_1 \rangle = \langle (1, p) \rangle \cong C_3 \oplus C_{2p}$.

If $H_E = \langle x_1, x_2 \rangle$ then $H_E^\perp = \langle y_1 \rangle = \langle (1, p) \rangle \cong C_3 \oplus C_{2p}$. We conclude that,

- (1) If $H_E = \{0\}$, then $H_E^\perp = D(L_{6,2p^2})$ and $D(L_{6,2p^2}/H_E^\perp) \cong \{0\}$.
- (2) If $H_E = \langle x_1 \rangle$, then $H_E^\perp = \langle (0, p) \rangle \cong C_{2p}$ and $D(L_{6,2p^2}/H_E^\perp) \cong C_6 \oplus C_p$.
- (3) If $H_E = \langle x_2 \rangle$, then $H_E^\perp = \langle (1, p) \rangle \cong C_3 \oplus C_{2p}$ and $D(L_{6,2p^2}/H_E^\perp) \cong C_2 \oplus C_p$.
- (4) If $H_E = \langle x_1, x_2 \rangle$, then $H_E^\perp = \langle (1, p) \rangle \cong C_3 \oplus C_{2p}$ and $D(L_{6,2p^2}/H_E^\perp) \cong C_2 \oplus C_p$.

The result follows. \square

It is likely that the following lemma was proved in [Bri83], but we prove it here as we were unable to locate a copy.

Lemma 3.3. *Let L be a lattice of signature $(2, n)$ and let $E \subset L$ be a primitive totally isotropic subspace of rank 2. If H_E is the subgroup given by $H_E = E_{L^\vee}^\perp$, then the discriminant form of the lattice E^\perp/E is given by*

$$D(E^\perp/E) \cong H_E^\perp/H_E \subset D(L).$$

Proof. Let $E \leq L$ be a primitive totally isotropic subspace of rank 2. As E and E^\perp are primitive in L , then as a \mathbb{Z} -module, $L \cong (E^\perp/E) \oplus E \oplus F$ for some $F \leq L$. As a \mathbb{Z} -module, $L^\vee = \text{Hom}(L, \mathbb{Z})$ assumes the following form

$$L^\vee \cong (E^\perp/E)^\vee \oplus (E \oplus F)^\vee$$

Moreover, the sublattice $E^{\perp\perp} \subset L^\vee$ is primitive in $(E \oplus F)^\vee$ and we can take a basis $\{e_1^*, f_1^*, e_2^*, f_2^*\}$ of $(E \oplus F)^\vee$ so that $E^{\perp\perp} = \langle e_1^*, e_2^* \rangle$, and such that the bilinear form on $(E \oplus F)^\vee \subset L^\vee$ is equal to $U \oplus U$. Because (E^\perp/E) is non-degenerate, $(E^\perp/E)^\vee$ has a basis \mathcal{B} in $(E^\perp/E) \otimes \mathbb{Q}$ and with respect to the basis $\{e_1^*, f_1^*, e_2^*, f_2^*\} \cup \mathcal{B}$, the form on L^\vee is $U \oplus U \oplus L_0$.

Therefore,

$$D(L) = L^\vee/L \cong \frac{\langle e_1^*, f_1^*, e_2^*, f_2^* \rangle}{E \oplus F} \oplus D(E^\perp/E).$$

As $H_E = \langle e_1^*, e_2^* \rangle/E$, we conclude that $D(E^\perp/E) \cong H_E^\perp/H_E \subset D(L)$. \square

Corollary 3.4. *Only the case $(a_1, a_1 a_2) = (1, 1)$ or $(1, 2p)$ occurs in Lemma 3.2.*

Proof. By Lemma 3.3, the negative definite lattice B has discriminant group $D(B) = H_E^\perp/H_E \leq D(L_{6,2p^2})$ and so if $(a_1, a_1 a_2) = (1, 6p)$, then $D(B) = ((1/2), C_2)$. By using tables in [CS99], we see that no such B can exist. The other cases may exist, though. If $(a_1, a_1 a_2) = (1, 2p)$, $D(B) = ((1/3), C_3)$ and $B = A_2(-1)$. If $(a_1, a_1 a_2) = (1, 1)$, $D(B) = ((-1/6) \oplus (-1/2p^2), C_6 \oplus C_{2p^2})$ and B may be equal to $\langle -6 \rangle \oplus \langle -2p^2 \rangle$. \square

Lemma 3.5. *There exists a basis $\{v_1, \dots, v_6\}$ for $L_{6,2p^2} \otimes \mathbb{Q}$ such that $\{v_1, v_2\}$ form a \mathbb{Z} -basis for E and $\{v_1, \dots, v_4\}$ form a \mathbb{Z} -basis for E^\perp and*

$$Q = ((v_i, v_j)) = \begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ A & 0 & 0 \end{pmatrix}$$

where A and B are as described previously in Lemma 3.2.

Proof. This is essentially Lemma 2.24 of [GHS07].

Let $R = -B^{-1}C \in M_2(\mathbb{Z}[1/\det B])$ and let $R' \in M_2(\mathbb{Z}[1/\det B])$ satisfy

$$D - {}^t C B^{-1} C + {}^t R' A + {}^t A R' = 0$$

and define the base change matrix

$$N = \begin{pmatrix} I & 0 & R' \\ 0 & I & R \\ 0 & 0 & I \end{pmatrix}.$$

□

Lemma 3.6. *The groups $N(F)$, $W(F)$ and $U(F)$ are given by*

$$\begin{aligned} N(F) &= \left\{ \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \mid \begin{array}{l} {}^tU AZ = A, {}^tX B X = B, {}^tV A Z = 0, {}^tX B Y + {}^tV A Z = 0 \\ {}^tY B Y + {}^tZ A W + {}^tW A Z = 0, \det(U) > 0 \end{array} \right\} \\ W(F) &= \left\{ \begin{pmatrix} I & V & W \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix} \mid B Y + {}^tV A = 0, {}^tY B Y + A W + {}^tW A = 0 \right\} \\ U(F) &= \left\{ \begin{pmatrix} I & 0 & \begin{pmatrix} 0 & a_1 a_2 x \\ -x & 0 \end{pmatrix} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \mid x \in \mathbb{R} \right\}. \end{aligned}$$

Proof. Direct calculation (as [Kon93]).

□

We also need a description for $N(F)_{\mathbb{Z}}$. As mentioned in Proposition 2.27 of [GHS07], if $g \in N(F)$ is given on the above basis then $g \in N(F)_{\mathbb{Z}}$ if

$$N^{-1}gN = \begin{pmatrix} U & V & -VB^{-1}C + W + UR' - R'Z \\ 0 & X & Y - XB^{-1}C + B^{-1}CZ \\ 0 & 0 & Z \end{pmatrix} \in \mathrm{GL}(6, \mathbb{Z}).$$

As in [Kon93] and [GHS07], we identify $D_L(F)$ with $(z, w_1, w_2, \tau) \in \mathbb{C} \times \mathbb{C}^2 \times \mathbb{H}$ as a Siegel domain. The identification proceeds by choosing homogeneous coordinates $[t_1 : \dots : t_6]$ on $\mathbb{P}(L \otimes \mathbb{C})$ and the map $\mathcal{D}_L(F) \rightarrow \mathbb{P}(L \otimes \mathbb{C})$ is defined by setting $t_6 = 1$ and

$$(6) \quad \begin{cases} t_1 \mapsto z \in \mathbb{C} \\ t_3 \mapsto w_1 \in \mathbb{C} \\ t_5 \mapsto \tau \\ t_2 \mapsto \frac{-2\delta z \tau - (w_1, w_2) B^t(w_1, w_2)}{2\delta a_2}. \end{cases}$$

Proposition 3.7. *Let*

$$g = \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \in N(F)$$

where $Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The action of g on $\mathcal{D}_L(F)$ is given by

$$\begin{cases} z \mapsto \frac{z}{\det Z} + (c\tau + d)^{-1} \left(\frac{c}{2\delta \det Z} {}^t \underline{w} B \underline{w} + \underline{V}_1 \underline{w} + W_{11} \tau + W_{12} \right) \\ \underline{w} \mapsto (c\tau + d)^{-1} (X \underline{w} + Y \begin{pmatrix} \tau \\ 1 \end{pmatrix}) \\ \tau \mapsto \frac{a\tau + b}{c\tau + d} \end{cases}$$

Proof. As in [GHS07]. \square

3.4. Counting the boundary components. We wish to examine the non-canonical singularities in X . As explained previously, the compactification may be chosen so that all the singularities at the 0 dimensional cusps are canonical. Therefore, we need only to consider the compactification at the 1 dimensional cusps. These correspond precisely to the Γ -orbits of totally isotropic planes in $L \otimes \mathbb{Q}$.

We begin by using the approach of [Sca87] to determine the $O(L_{6,2})$ -orbits of totally isotropic planes in $L_{6,2} \otimes \mathbb{Q} = L_{6,2p^2} \otimes \mathbb{Q}$. This involves showing that given a totally isotropic subspace $E \leq L_{6,2}$, the bilinear form on $L_{6,2}$ can be put into a certain normal form.

Lemma 3.8. *If $E \leq L_{6,2}$ is primitive and totally isotropic of rank 2, then $E^\perp/E \cong \langle -6 \rangle \oplus \langle -2 \rangle$ or $E^\perp/E \cong A_2(-1)$.*

Proof. We consider the subspaces $H_E \leq D(L_{6,2})$. As E is totally isotropic, $H_E \leq D(L_{6,2})$ is totally isotropic. As usual, identify $D(L_{6,2})$ with $C_6 \oplus C_2$. If $(a, b) \in D(L_{6,2})$ is isotropic, then $a^2/6 - b^2/2 = 0 \pmod{\mathbb{Z}}$. The solutions of which are given by $(a, b) = (0, 0)$ and $(a, b) = (3, 1)$. If $H_E = \{(0, 0)\}$, then $H_E^\perp/H_E = D(L_{6,2})$ with form $((-1/6) \oplus (1/2), C_6 \oplus C_2)$. If $H_E = \langle (3, 1) \rangle$, then $H_E^\perp = \langle (1, 1) \rangle$ and $H_E^\perp/H_E \cong \langle (2, 0) \rangle$ with form $((1/3), C_3)$. By using tables in [CS99], we see that there are two negative definite even lattices of determinant 12: $\langle -6 \rangle \oplus \langle -2 \rangle$ and $\begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix}$ but, as calculated previously, the discriminant form of the second lattice is inequivalent to $((1/2)^{\oplus 2} \oplus (-1/3), C_2^{\oplus 2} \oplus C_3)$. Therefore, in the case $H_E = \langle (0, 0) \rangle$ we have $E^\perp/E \cong \langle -6 \rangle \oplus \langle -2 \rangle$; similarly, by using tables in [CS99], in the case $H_E = \langle (3, 1) \rangle$ we have $E^\perp/E \cong A_2(-1)$. \square

Lemma 3.9. *Let $E \leq L_{6,2} \otimes \mathbb{Q}$ be a totally isotropic subspace of rank 2. Then there exists a \mathbb{Z} -basis $\{v_1, \dots, v_6\}$ of $L_{6,2}$ such that $\{v_1, v_2\}$ is a basis for E and $\{v_1, \dots, v_4\}$ is a basis for E^\perp and the inner product on $L_{6,2}$ becomes*

$$Q = ((v_i, v_j)) = \begin{pmatrix} 0 & 0 & P \\ 0 & B & C \\ P & {}^t C & Q \end{pmatrix}$$

where

- (1) If $H_E = \langle (1, 1) \rangle$, then $B = \langle -6 \rangle \oplus \langle -2 \rangle$ and $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Q = C = 0$.
- (2) If $H_E = \langle (3, 1) \rangle$, then $B = A_2(-1)$ and $P = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 2d & 0 \\ 0 & 0 \end{pmatrix}$ for $d \in \{0, 1, 2\}$ and $C = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ for $c \in \{0, 1, 2\}$.

Proof. We start by taking a basis $\{v_1, \dots, v_6\}$ of $L_{6,2}$ for which $\{v_1, v_2\}$ is a basis for E and $\{v_1, \dots, v_4\}$ is a basis for E^\perp . Suppose that on this basis

$$Q = ((v_i, v_j)) = \begin{pmatrix} 0 & 0 & A_0 \\ 0 & B_0 & C_0 \\ {}^t A_0 & {}^t C_0 & D_0 \end{pmatrix}.$$

By Lemma 3.8, $H_E = \langle (0, 0) \rangle$ or $H_E = \langle (3, 1) \rangle$. If $H_E = \langle (0, 0) \rangle$ then, by the Elementary Divisor Theorem, there exist $U, Z \in \text{GL}(2, \mathbb{Z})$ such that

$$U A_0 Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Moreover, there exists $X \in \text{GL}(2, \mathbb{Z})$ such that ${}^tXB_0X = B = \langle -6 \rangle \oplus \langle -2 \rangle$, and so the matrix g_1 given by $g_1 = \text{diag}(U, X, Z) \in \text{GL}(6, \mathbb{Z})$ transforms Q to Q' where

$$Q' = {}^tg_1Qg_1 = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C_1 \\ {}^tA & {}^tC_1 & D_1 \end{pmatrix}.$$

Now consider the matrix g_2 given by

$$g_2 = \begin{pmatrix} I & -{}^tA{}^tC_1 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \in \text{GL}(6, \mathbb{Z}).$$

The map g_2 transforms Q' to Q'' where

$$Q'' = \begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ {}^tA & 0 & D_2 \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We next require that D_2 be put into the correct form. Consider the matrix g_3 given by

$$g_3 = \begin{pmatrix} I & 0 & W \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \in \text{GL}(6, \mathbb{Z})$$

g_3 sends $D_2 \mapsto D_2 + {}^tWA + {}^tAW$. One checks that ${}^tWA + {}^tAW$ contains all matrices of the form

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

where $a, b, c \in \mathbb{Z}$. Therefore, there exists W so that g_3 sends

$$D_2 \mapsto \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}$$

where d_{11} and d_{22} are taken modulo 2. However, as the form Q is even, d_{11} and d_{22} are both even. Therefore, there exists W so that g_3 sends D_2 to 0. The matrix $g_3g_2g_1 \in \text{GL}(6, \mathbb{Z})$ gives the required base change.

If $H_E = \langle (3, 1) \rangle$ then, by the Elementary Divisor Theorem, there exist $U, Z \in \text{GL}(2, \mathbb{Z})$ such that

$$UA_0Z = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$$

Moreover, there exists $X \in \text{GL}(2, \mathbb{Z})$ such that ${}^tXB_0X = B = A_2(-1)$, and so the matrix g_4 given by $g_4 = \text{diag}(U, X, Z) \in \text{GL}(2, \mathbb{Z})$ transforms Q to Q' where

$$Q' = {}^tg_4Qg_4 = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C_1 \\ {}^tA & {}^tC_1 & D_1 \end{pmatrix}.$$

and

$$A = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}.$$

Now consider the matrix g_5 given by

$$g_5 = \begin{pmatrix} I & P & 0 \\ 0 & I & Q \\ 0 & 0 & I \end{pmatrix} \in \mathrm{GL}(6, \mathbb{Z}).$$

for some $P, Q \in M_2(\mathbb{Z})$. We claim that P and Q can be chosen such that

$${}^tPA + BQ + C_1 = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$$

where a is determined modulo 3. We have

$${}^tPA + BQ + C_1 = \begin{pmatrix} 3p_{21} - 2q_{11} - q_{21} + c_{11} & p_{11} - 2q_{12} + q_{22} + c_{12} \\ 3p_{22} - 2q_{21} - q_{11} + c_{21} & p_{12} - q_{12} - 2q_{22} + c_{22} \end{pmatrix}$$

The claim about the second column is immediate as p_{11} and p_{12} are both free. For the first column, we can work modulo 3 as p_{21} and p_{22} are free. As δ defined by

$$\delta = 2q_{11} + q_{21} = -(2q_{21} + q_{11}) \pmod{3}$$

the first column can be mapped to ${}^t(0, c_{11} + c_{21})$ modulo 3 for an appropriate choice of δ . Therefore,

$$g_5 = \begin{pmatrix} I & P & 0 \\ 0 & I & Q \\ 0 & 0 & I \end{pmatrix}$$

with P and Q chosen as above transforms Q' to

$$Q'' = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C_0 \\ {}^tA & {}^tC_0 & D_2 \end{pmatrix}$$

where C_0 is as in the statement of the theorem. We next require that D_2 be put into the correct form. Consider the matrix g_6 defined by

$$g_6 = \begin{pmatrix} I & 0 & W \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \in \mathrm{GL}(6, \mathbb{Z})$$

for $W \in M_2(\mathbb{Z})$. The matrix g_6 sends

$$D_2 \mapsto D_2 + {}^tWA + {}^tAW.$$

One checks that the set $\{{}^tWA + {}^tAW \mid W \in M_2(\mathbb{Z})\}$ contains all matrices of the form

$$\begin{pmatrix} 6a & b \\ b & 2c \end{pmatrix}$$

where $a, b, c \in \mathbb{Z}$. Therefore, there exists W so that g_3 sends

$$D_2 \mapsto \begin{pmatrix} d_{11} & 0 \\ 0 & d_{22} \end{pmatrix}$$

where d_{11} is taken modulo 6 and d_{22} is taken modulo 2. As the form Q is even, d_{11} and d_{22} are both even and therefore there exists W so that g_3 sends d_{11} to one of 0, 2 or 4 and the rest to 0. Therefore $g_6g_5g_3 \in \mathrm{GL}(6, \mathbb{Z})$ gives the required base change. \square

Theorem 3.10. *The modular variety \mathcal{F}_Γ has at most $320(p^5 + p^2)$ rank 2 boundary components.*

Proof. If $E_1, E_2 \leq L_{6,2}$ are primitive totally isotropic subspaces of rank 2 with the same normal form, then by Lemma 3.9, there exist bases $\{v_1, \dots, v_6\}$ and $\{w_1, \dots, w_6\}$ of $L_{6,2}$ such that $\{v_1, v_2\}, \{w_1, w_2\}$ are bases for E_1 and E_2 respectively and

$$((v_i, v_i)) = ((w_i, w_j)) = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C \\ A & {}^t C & D \end{pmatrix}.$$

Accordingly, one can define $g \in \mathrm{O}(L_{6,2})$ by $g : v_i \mapsto w_i$ such that $g(E_1) = E_2$ and so there are at most 20 totally isotropic rank 2 subspaces of $L_{6,2}$ up to $\mathrm{O}^+(L_{6,2})$ equivalence. By Theorem 2.12,

$$|\mathrm{O}^+(L_{6,2}) : \mathrm{O}^+(L_6, h_{2p^2}^s)| = 16(p^5 + p^2)$$

and so, up to $\mathrm{O}^+(L_6, h_{2p^2}^s)$ equivalence, there are at most $320(p^5 + p^2)$ rank 2 boundary components. \square

3.5. Counting the singularities. We next show that the set of fixed points can be reduced by application of special elements in $N(F)_\mathbb{Z}$. This enables us to produce an upper bound for the number of components of the singular locus.

For a given boundary component F , we define $N = a_1 a_2 \det B$.

Lemma 3.11. *Let E be a rank 2 totally isotropic subspace corresponding to the boundary component F . Let $A = \mathrm{diag}(a_1, a_1 a_2)$, as in Lemma 3.5. Then the principal congruence subgroup of level N , $\Gamma(N)$, embeds in $N(F)$. The embedding is given by sending $Z \in \Gamma(N)$ to*

$$g_Z = \begin{pmatrix} Z' & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Z \end{pmatrix} \in N(F)_\mathbb{Z}$$

where, if

$$Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{we write} \quad Z' = \begin{pmatrix} d & -ca_2 \\ -b/a_2 & a \end{pmatrix}.$$

Proof. Let

$$g = \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \in N(F).$$

If $g \in N(F)_\mathbb{Z}$ then, by Lemma 3.6, the following are integral matrices

$$\begin{aligned} (7) \quad & U = X = Z \\ (8) \quad & -VB^{-1}C + W + UR' - RZ \\ (9) \quad & Y - XB^{-1}C + B^{-1}CZ. \end{aligned}$$

Let

$$Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$$

and let $X = I$ and $V = W = Y = 0$. By Lemma 3.5, we can suppose that $R', B^{-1} \in M_2(\mathbb{Z}[1/\det B])$. By Lemma 3.6, ${}^tUAZ = A$ and so

$$U = \begin{pmatrix} d & -ca_2 \\ -b/a_2 & a \end{pmatrix}.$$

As $Z \in \Gamma(N)$, it follows that $U \in M_2(\mathbb{Z})$. Because of Equations (8) and (9), we obtain the following integral matrices:

$$(10) \quad UR' - R'Z$$

$$(11) \quad -B^{-1}C + B^{-1}CZ.$$

If

$$R' = \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

then

$$UR' - R'Z = \begin{pmatrix} -a_2cy - aw + dw - cx & -a_2cz - bw \\ -cz - bw/a_2 & -by + az - dz - bx/a_2 \end{pmatrix} \in M_2(\mathbb{Z}).$$

As $Z \in \Gamma(N)$, then $a \equiv d \equiv 1 \pmod{N}$ and $b \equiv c \equiv 0 \pmod{N}$ and so Equation (3.5) is satisfied. Furthermore, $Z \equiv I \pmod{N}$ and so $C - CZ \equiv 0 \pmod{N}$. As $\det B|N$,

$$-B^{-1}C + B^{-1}CZ = B^{-1}(C - CZ) \in M_2(\mathbb{Z})$$

and so $\Gamma(N) \leq N(F)_{\mathbb{Z}}$. □

Lemma 3.12. *Let E be a rank 2 totally isotropic subspace corresponding to the boundary component F . Let $A = \text{diag}(a_1, a_1a_2)$, as in Lemma 3.5. The group $W(F)_{\mathbb{Z}}$ contains all elements of the form*

$$g_Y = \begin{pmatrix} I & * & * \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix}$$

where $Y \in M_2(N\mathbb{Z})$.

Proof. If

$$g_Y = \begin{pmatrix} I & V & W \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix} \in W(F)_{\mathbb{Z}}$$

then by Lemma 3.6,

$$(12) \quad BY + {}^tVA = 0$$

$$(13) \quad {}^tYBY + AW + {}^tWA = 0.$$

Furthermore, by Lemma 3.5,

$$N^{-1}gN = \begin{pmatrix} I & V & W - VB^{-1}C \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix}$$

subject to the conditions that

$$(14) \quad W - VB^{-1}C$$

$$(15) \quad V = Y = 0$$

are both integral. We look for solutions satisfying Equation (15). Equation (12) has a solution in V if $Y \in M_2(a_1 a_2 \mathbb{Z})$ and Equation (14) is satisfied if $V \in M_2(\det B \mathbb{Z})$. Because of Equation (12), we can ensure that both are satisfied if $Y \in M_2(N \mathbb{Z})$.

If

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

then Equation (13) becomes

$$\begin{aligned} -{}^t Y B Y &= A W + {}^t W A \\ &= \begin{pmatrix} 2a_1 w_{11} & a_1 w_{12} + a_1 a_2 w_{12} \\ a_1 a_2 w_{21} + a_1 w_{12} & 2a_1 a_2 w_{22} \end{pmatrix} \end{aligned}$$

and, by considering Equation (12), has a solution in W if $Y \in M_2(2a_1 a_2 \mathbb{Z})$. All such conditions are clearly satisfied if $Y \in M_2(N \mathbb{Z})$. \square

Theorem 3.13. *If $(a_1, a_1 a_2) = (1, 1)$ the singular locus of a boundary component contains of the order of p^6 points and p^5 lines. The number of surfaces in the boundary component does not depend on p . If $(a_1, a_1 a_2) = (1, 2p)$ the singular locus of a boundary component contains of the order of p^{14} points, p^{12} lines, and p^9 surfaces.*

Proof. By Proposition 3.7, g acts on (z, \underline{w}, τ) by

$$\begin{aligned} z &\mapsto \frac{z}{\det Z} + (c\tau + d)^{-1} \left(\frac{c}{2\delta \det Z} {}^t \underline{w} B \underline{w} + \underline{V}_1 \underline{w} + W_{11} \tau + W_{12} \right) \\ \underline{w} &\mapsto (c\tau + d)^{-1} (X \underline{w} + Y \begin{pmatrix} \tau \\ 1 \end{pmatrix}) \\ \tau &\mapsto \frac{a\tau + b}{c\tau + d}. \end{aligned}$$

In particular (as noted in [GHS07]), τ is $\text{SL}(2, \mathbb{Z})$ equivalent to i or a cube root of unity ω . Indeed, $\tau \in \text{SL}(2, \mathbb{Z})i$ if Z is of order 4 and $\tau \in \text{SL}(2, \mathbb{Z})\xi_3$ if Z is of order 3 or 6. Moreover, if

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$$

then

$$\tau = \frac{\alpha\theta + \beta}{\gamma\theta + \delta}$$

where $\theta \in \{i, \xi_3\}$ and so

$$\tau = \frac{(\alpha\gamma + \delta\beta) + (\alpha\delta + \beta\gamma) \text{Re } \theta + (\alpha\delta - \beta\gamma) \text{Im } \theta i}{\gamma^2 + \delta^2 + 2\gamma\delta \text{Re } \theta}.$$

and we define J by

$$J = 2(\gamma^2 + \delta^2 + 2\gamma\delta \text{Re } \theta)$$

and K_1 and K_2 by

$$\tau = \frac{K_1}{J} + \frac{K_2}{J} v,$$

where $v \in \{i, \omega\}$. At \underline{w} ,

$$(16) \quad \underline{w} = (c\tau + d)^{-1} (X \underline{w} + Y \begin{pmatrix} \tau \\ 1 \end{pmatrix}).$$

For Z defined by g , we define $\xi = (c\tau + d)^{-1}$ and T by

$$T = I - \xi X.$$

As observed in [GHS07] Proposition 2.28, ξ is a sixth or a fourth root of unity. This follows because $G_4(i) \neq 0$ and $G_6(\xi_3) \neq 0$ where G_k is the weight- k Eisenstein series (see [DS05]). In particular, ξ is a sixth root of unity if Z is of order 3 or 6 and a fourth root of unity if Z is of order 4.

If $\det T \neq 0$, then

$$\underline{w} \in T^{-1}Y \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

and so, by noting that $Y \in M_2(\mathbb{Z}[1/\det B])$, we have that $\underline{w} \in L \times L$ where

$$L = \frac{\langle 1, \tau \rangle}{\det T \det B}$$

(and where $\langle 1, \tau \rangle$ denotes the lattice in \mathbb{C} generated by 1 and τ). We can assume that the basis $\{v_1, \dots, v_6\}$ is given so that $\{v_3, v_4\}$ defines the fundamental polyhedron of the lattice B . By the crystallographic restriction theorem (p. 50 of [Sen95]), if $g \in O(B)$ then g has order 1, 2, 3, 4 or 6 and B admits a basis such that g is given by $\pm I_2$ or by

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \quad \text{if} \quad \chi_g(x) = \phi_1(x)\phi_2(x) \\ \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}^{\pm 1} & \quad \text{if} \quad \chi_g(x) = \phi_3(x) \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\pm 1} & \quad \text{if} \quad \chi_g(x) = \phi_4(x) \\ \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^{\pm 1} & \quad \text{if} \quad \chi_g(x) = \phi_6(x). \end{aligned}$$

We shall therefore assume that B and X are given on the above basis. The value of $\det B$ in each case is given in Table 1.

ξ	i	-1	$-i$	1	ξ_6	ω	ω^2	ξ_6^5
$\phi_1(x)^2$	$2i$	4	$-2i$	0	$\xi_6 - 1$	$3\xi_6$	$-3\xi_6 + 3$	$-\xi_6$
$\phi_1(x)\phi_2(x)$	2	0	2	0	$\xi_6 + 1$	ξ_3^2	$\xi_6 + 1$	ξ_3^2
$\phi_2(x)^2$	$-2i$	0	$2i$	4	$-3\xi_6 + 3$	$-\xi_6$	$\xi_6 - 1$	$3\xi_6$
$\phi_3(x)$	$-i$	1	i	3	$-2\xi_6 + 2$	0	0	$2\xi_6$
$\phi_4(x)$	0	2	0	2	$-\xi_6 + 1$	ξ_6	$-\xi_6 + 1$	ξ_6
$\phi_6(x)$	i	3	$-i$	1	0	$2\xi_6$	$-2\xi_6 + 2$	0

TABLE 1.

We next consider L for each value of $\det B$. By direct calculation we find that,

$$\begin{aligned} \text{If } Z \text{ is order 4,} \quad & L \leq \frac{\langle 1, i \rangle}{JK \det B} \quad K = 1, 2, 3, 4 \\ \text{if } Z \text{ is order 3 or 6,} \quad & L \leq \frac{\langle 1, \sqrt{3}i \rangle}{2KJ \det B} \quad K = 1, 2, 3, 4, 6. \end{aligned}$$

We next bound the number of components of the singular locus in each boundary component by using the elements defined in Lemma 3.11 and Lemma 3.12.

By Lemma 3.11, $\Gamma(N) \leq N(F)$. It is well known (see [DS05]) that

$$|\mathrm{SL}(2, \mathbb{Z}) : \Gamma(N)| = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$$

and as

$$|\mathrm{O}^+(L_{6,2p^2}) : \widetilde{\mathrm{O}}^+(L_{6,2p^2})| = 16,$$

there are at most K_N equivalence classes of τ modulo $N(F) \cap \mathrm{O}(L_6, h_{2p^2}^s)$ where K_N is defined by

$$K_N = 16N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

If Z is of order 4, then

$$w_j = \frac{x_{1j}}{JK \det B} + \frac{x_{2j}i}{JK \det B} \in \frac{\langle 1, i \rangle}{JK \det B}$$

and

$$g_Y : w_j \mapsto \frac{x_{1j} + KK_1 \det BY_{j1} + Y_{j2} JK \det B}{JK \det B} + \frac{(x_{2j} + K \det BK_2 Y_{j1})i}{JK \det B}$$

and as $Y \in M_2(N\mathbb{Z})$ can be chosen arbitrarily, x_{1j} can be reduced modulo $NJK \det B$ and x_{2j} can be reduced modulo $NKK_2 \det B$.

If Z is of order 3 or 6, then

$$w_j = \frac{x_{1j}}{2JK \det B} + \frac{x_{2j}\sqrt{3}i}{2JK \det B} \in \frac{\langle 1, \sqrt{3}i \rangle}{2JK \det B}$$

and

$$g_Y : w_j \mapsto \frac{x_{1j} + 2KK_1 \det BY_{j1} + 2Y_{j2} JK \det B}{JK \det B} + \frac{(x_{2j} + 2K \det BK_2 Y_{j1})\sqrt{3}i}{2JK \det B}$$

and as $Y \in M_2(N\mathbb{Z})$ can be chosen arbitrarily, x_{1j} can be reduced modulo $2NJK \det B$ and x_{2j} can be reduced modulo $2NKK_2 \det B$. We consider the cases where $\det T = 0$ separately. They occur when

$(\chi_X(x), \xi) \in \{(\phi_1\phi_2, -1), (\phi_2^2, -1), (\phi_4, -i), (\phi_2^2, 1), (\phi_1\phi_2, 1), (\phi_6, \xi_6), (\phi_3, \xi_3), (\phi_3, \xi_3^2), (\phi_6, \xi_6^5)\}$. In each case we solve Equation (16) directly, and reduce as above. We find that:

- (1) If $(\chi_X(x), \xi) = (\phi_1\phi_2, -1)$, $w_2 \in \mathbb{C}$ is free and $w_1 \in \frac{\langle 1, i \rangle}{2J \det B}$ and so w_1 can be reduced to one of $2NJ \det B$ points.
- (2) If $(\chi_X(x), \xi) = (\phi_2^2, -1)$, $w_1, w_2 \in \mathbb{C}$ are free.
- (3) If $(\chi_X(x), \xi) = (\phi_4, -i)$, $w_2 \in \mathbb{C}$ is free and $w_1 = iw_2 + x_1$ for $x_1 \in w_1 \in \frac{\langle 1, i \rangle}{J \det B}$ and so w_1 can be reduced to one of $NJ \det B$ lines.
- (4) If $(\chi_X(x), \xi) = (\phi_2^2, 1)$, $w_1, w_2 \in \frac{\langle 1, i \rangle}{J \det B}$ or $w_1, w_2 \in \frac{\langle 1, \sqrt{3}i \rangle}{2J \det B}$ and so each of w_1, w_2 can be reduced to one of $2NJ \det B$ points.
- (5) If $(\chi_X(x), \xi) = (\phi_1\phi_2, 1)$, $w_1 \in \mathbb{C}$ is free and $w_2 \in \frac{\langle 1, i \rangle}{2J \det B}$ or $w_2 \in \frac{\langle 1, \sqrt{3}i \rangle}{4J \det B}$ and so w_1 can be reduced to one of $4NJ \det B$ points.

- (6) If $(\chi_X(x), \xi) = (\phi_6, \xi_6)$, $w_2 \in \mathbb{C}$ is free, $w_1 = -\xi_6 + x_1$ for $x_1 \in \frac{\langle 1, \sqrt{3}i \rangle}{2J \det B}$ and so w_1 can be reduced to one of $2NJ \det B$ points.
- (7) If $(\chi_X(x), \xi) = (\phi_3, \xi_3)$, $w_2 \in \mathbb{C}$ is free and $w_1 = \xi_3 + x_1$ for $x_1 \in \frac{\langle 1, \sqrt{3}i \rangle}{2J \det B}$ and so w_1 can be reduced to one of $2NJ \det B$ points.
- (8) If $(\chi_X(x), \xi) = (\phi_3, \xi_3^2)$, $w_2 \in \mathbb{C}$ is free and $w_1 = \xi_3 + x_1$ for $x_1 \in \frac{\langle 1, \sqrt{3}i \rangle}{2J \det B}$ and so w_1 can be reduced to one of $2NJ \det B$ points.
- (9) If $(\chi_X(x), \xi) = (\phi_6, \xi_6^5)$, $w_2 \in \mathbb{C}$ and $w_1 = x_1 + \xi_6 w_2$ for $x_1 \in \frac{\langle 1, \sqrt{3}i \rangle}{2J \det B}$ and so w_1 can be reduced to one of $2NJ \det B$ points.

After reduction by suitable $g_Y g_Z \in N(F) \cap O^+(L_6, h_{2p^2}^s)$, we conclude that the singular locus of each boundary component consists of at most $96K_N N^2 JK^2 K_2 \det B + 14K_N NJ \det B$ points; $K_N NJ \det B$ lines; and K_N surfaces.

We have at once that $|J| \leq 3N^3$ and $K \leq 6$. By Corollary 3.4,

$$\det B = \begin{cases} 12p^2 & \text{if } (a_1, a_1 a_2) = (1, 1) \\ 3 & \text{if } (a_1, a_1 a_2) = (1, 2p) \end{cases}$$

and one checks that

$$K_N = \begin{cases} 24 & \text{if } (a_1, a_1 a_2) = (1, 1) \\ 9216p^7(p^2 - 1) & \text{if } (a_1, a_1 a_2) = (1, 2p) \end{cases}$$

and so

$$96K_N N^2 JK^2 K_2 \det B + 14K_N NJ \det B = \begin{cases} o(p^6) & \text{if } (a_1, a_1 a_2) = (1, 1) \\ o(p^{14}) & \text{if } (a_1, a_1 a_2) = (1, 2p) \end{cases}$$

and

$$K_N NJ \det B = \begin{cases} o(p^5) & \text{if } (a_1, a_1 a_2) = (1, 1) \\ o(p^{12}) & \text{if } (a_1, a_1 a_2) = (1, 2p). \end{cases}$$

In each case, a sharp bound can be given. □

We end by remarking that as in [Kon93] and [GHS07], the action of

$$g = \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \in N(F)$$

on the tangent space is given by

$$\begin{pmatrix} \exp_{a_2}(t) & 0 & 0 \\ * & (c\tau + d)^{-1}X & 0 \\ * & * & (c\tau + d)^{-2} \end{pmatrix}.$$

Here,

$$t = (c\tau + d)^{-1} \left(\frac{c}{2\delta \det Z} {}^t \underline{w} B \underline{w} + c {}^t \underline{w} B \underline{w} / 2a_1 + \underline{V}_1 \underline{w} + W_{11}\tau + W_{12} \right)$$

and is, of course, equal to 0 at each boundary component. We wish to examine the singularities at the fixed points. As they are finite quotient singularities, one can use the *Reid-Tai criterion* to establish which are canonical. A special role is played by

quasi-reflections. These are elements $g \in \mathrm{GL}(n, \mathbb{C})$ such that $g \sim \mathrm{diag}(1, \dots, 1, \zeta)$ and $\zeta \neq 1$.

Definition 3.14. *The Reid-Tai sum $\Sigma(g)$ of $g \in \mathrm{GL}(n, \mathbb{C})$ is of finite order $m > 1$ is given by*

$$\Sigma(g) = \sum_{i=1}^m \left\{ \frac{a_i}{m} \right\}$$

where $\zeta^{a_1}, \dots, \zeta^{a_m}$ are the eigenvalues of g for $\zeta = e^{2\pi i/m}$, and $0 \leq \{x\} < 1$ denotes the fractional part of x . If $g = \mathrm{id}$, $\Sigma(g) := 1$.

The Reid-Tai criterion states that

Theorem 3.15. *[GHS13] if $G \leq \mathrm{GL}(n, \mathbb{C})$ is a finite subgroup not containing quasi-reflections, then \mathbb{C}^n/G is non-canonical if and only if*

$$\Sigma(g) \geq 1$$

for all $g \in G$.

If G contains quasi-reflections, there is the following modified version of the Reid-Tai criterion due to Katharina Ludwig [GHS13].

Definition 3.16. *If $g \in \mathrm{GL}(n, \mathbb{C})$ is of finite order $m > 1$, let $k \in \mathbb{N}_0$ be minimal with the property that g^k is a quasi-reflection or the identity. Let s be such that $m = sk$ and let g have eigenvalues $\zeta^{a_1}, \dots, \zeta^{a_m}$ for $\zeta = e^{2\pi i/m}$ where $\{a_i\}$ are ordered so that $\zeta^{ka_1} = \zeta^{ka_{n-1}} = 1$. The modified Reid-Tai sum $\Sigma'(g)$ is defined by*

$$\Sigma'(g) = \left\{ \frac{sa_n}{m} \right\} + \sum \left\{ \frac{a_i}{m} \right\}$$

and $\Sigma'(1) := 1$. (Note that $\Sigma'(g) = \Sigma(g)$ if no power of g is a quasi-reflection.)

Proposition 3.17. *(Proposition 5.24 [GHS13]) If G is a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$. Then \mathbb{C}^n/G has canonical singularities if $\Sigma'(g) > 1$ for all $g \in G$.*

And so, by using the classification of the elliptic elements of $\mathrm{SL}(2, \mathbb{Z})$ and the crystallographic restriction theorem for B , we find that if \mathbb{C}^4/G is a singularity at the boundary and $g \in G$ is such that $\mathbb{C}^4/\langle g \rangle$ is non-canonical, then $\mathbb{C}^4/\langle g \rangle$ is either $\frac{1}{3}(3, 3, 1, 1)$ or $\frac{1}{6}(6, 2, 1, 1)$.

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